

## THE PENTAGONAL FUZZY NUMBERS

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**ABSTRACT.** We define the pentagonal fuzzy sets and generalize the results of addition, subtraction, multiplication, and division based on the Zadeh's extension principle for two pentagonal fuzzy sets. In addition, we find the condition that the result of addition or subtraction for two pentagonal fuzzy sets becomes a triangular fuzzy number and give some example.

### 1. Introduction

Dubois and Prade defined the extended algebraic operations by applying the extension principle to normal algebraic operations, which are addition, subtraction, multiplication and division. And they get some results on the calculations of these operations for fuzzy triangular numbers. The extension principle is one of the most basic concepts of fuzzy set theory which can be used to generalize crisp mathematical concepts of fuzzy sets. We have already found so many results of four operations based on Zadeh's extension principle for two generalized triangular fuzzy sets, quadratic fuzzy numbers, generalized quadratic fuzzy sets, generalized trapezoidal fuzzy sets and quadrangular fuzzy sets([1, 2, 3]).

In this paper, we derive the results of operations for two pentagonal fuzzy numbers and find the conditions in which the result of addition or subtraction for two pentagonal fuzzy numbers becomes a quadrangular or triangular fuzzy numbers.

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## 2. Preliminaries

The extension principle is defined as follows:

DEFINITION 2.1. ([4]) Let  $X = X_1 \times \cdots \times X_n$  be a cartesian product and  $\mu_i$  be a fuzzy set in  $X_i$ , respectively, and  $f : X \rightarrow Y$  be a mapping. Then the extension principle allows us to define a fuzzy set  $\nu$  in  $Y$  by

$$\nu(y) = \begin{cases} \sup_{(x_1, \dots, x_n) \in f^{-1}(y)} \min\{\mu_1(x_1), \dots, \mu_n(x_n)\} & (f^{-1}(y) \neq \emptyset) \\ 0 & (f^{-1}(y) = \emptyset). \end{cases}$$

For  $n = 1$ , the extension principle reduces to a fuzzy set  $\nu = f(\mu)$  defined by

$$\nu(y) = \begin{cases} \sup_{x \in f^{-1}(y)} \mu(x) & (f^{-1}(y) \neq \emptyset) \\ 0 & (f^{-1}(y) = \emptyset). \end{cases}$$

Applying the extension principle to algebraic operations for fuzzy sets, we have the following definitions for extended operations.

DEFINITION 2.2. ([4]) The extended addition, extended subtraction, extended multiplication, and extended division are defined by

(1) Addition  $A(+ )B$  :

$$\mu_{A(+ )B}(z) = \sup_{z=x+y} \min\{\mu_A(x), \mu_B(y)\}, \quad x \in A, y \in B.$$

(2) Subtraction  $A(- )B$  :

$$\mu_{A(- )B}(z) = \sup_{z=x-y} \min\{\mu_A(x), \mu_B(y)\}, \quad x \in A, y \in B.$$

(3) Multiplication  $A(\cdot )B$  :

$$\mu_{A(\cdot )B}(z) = \sup_{z=x \cdot y} \min\{\mu_A(x), \mu_B(y)\}, \quad x \in A, y \in B.$$

(4) Division  $A(/ )B$  :

$$\mu_{A(/ )B}(z) = \sup_{z=x/y} \min\{\mu_A(x), \mu_B(y)\}, \quad x \in A, y \in B.$$

For the calculations, we use the concepts of  $\alpha$ -cut.

DEFINITION 2.3. An  $\alpha$ -cut of the fuzzy number  $A$  is defined by  $A_\alpha = \{x \in \mathbb{R} \mid \mu_A(x) \geq \alpha\}$  if  $\alpha \in (0, 1]$  and  $A_\alpha = \text{cl}\{x \in \mathbb{R} \mid \mu_A(x) > \alpha\}$  if  $\alpha = 0$ .

The following definition of a triangular fuzzy number and a left(right) quadrangular fuzzy set is already used in a few of papers([2], [3]).

DEFINITION 2.4. A *triangular fuzzy number* is a fuzzy set  $A = (a_1, a_2, a_3)$  having membership function

$$\mu_A(x) = \begin{cases} 0, & x < a_1, a_3 \leq x \\ \frac{x-a_1}{a_2-a_1}, & a_1 \leq x < a_2 \\ \frac{a_3-x}{a_3-a_2}, & a_2 \leq x < a_3 \end{cases}$$

DEFINITION 2.5. A left quadrangular fuzzy set is a fuzzy set  $A$  having membership function

$$\mu_A(x) = \begin{cases} 0, & x < a_1, a_4 \leq x \\ \frac{x-a_1}{2(a_2-a_1)}, & a_1 \leq x < a_2 \\ \frac{x-a_3}{2(a_3-a_2)} + 1, & a_2 \leq x < a_3 \\ \frac{a_4-x}{a_4-a_3}, & a_3 \leq x < a_4 \end{cases}$$

where  $a_i \in \mathbb{R}, i = 1, 2, 3, 4$  and  $a_1 + a_3 > 2a_2$ . This left quadrangular fuzzy set is denoted by  $A = (a_1, a_2^*, a_3, a_4)$ .

DEFINITION 2.6. A right quadrangular fuzzy set is a fuzzy set  $A$  having membership function

$$\mu_A(x) = \begin{cases} 0, & x < a_1, a_4 \leq x \\ \frac{x-a_1}{a_2-a_1}, & a_1 \leq x < a_2 \\ \frac{a_2-x}{2(a_3-a_2)} + 1, & a_2 \leq x < a_3 \\ \frac{a_4-x}{2(a_4-a_3)}, & a_3 \leq x < a_4 \end{cases}$$

where  $a_i \in \mathbb{R}, i = 1, 2, 3, 4$  and  $a_2 + a_4 < 2a_3$ . This right quadrangular fuzzy set is denoted by  $A = (a_1, a_2, a_3^*, a_4)$ .

REMARK 2.7. ([2]) In order for a left(right) quadrangular  $A$  to be a triangular fuzzy number, it is necessary that  $2a_2 = a_1 + a_3$  ( $2a_3 = a_2 + a_4$ ) should be met.

### 3. The pentagonal fuzzy numbers

We define a pentagonal fuzzy number to generalize the results of this paper as follows.

DEFINITION 3.1. A *pentagonal fuzzy number* is a fuzzy set  $A$  having membership function

$$\mu_A(x) = \begin{cases} 0, & x < a_1, a_5 \leq x \\ \frac{x-a_1}{2(a_2-a_1)}, & a_1 \leq x < a_2 \\ \frac{x-a_3}{2(a_3-a_2)} + 1, & a_2 \leq x < a_3 \\ \frac{a_3-x}{2(a_4-a_3)} + 1, & a_3 \leq x < a_4 \\ \frac{a_5-x}{2(a_5-a_4)}, & a_4 \leq x < a_5 \end{cases}$$

where  $a_i \in \mathbb{R}, i = 1, 2, 3, 4, 5$ ,  $a_1 + a_3 > 2a_2$ , and  $a_3 + a_5 < 2a_4$ . This pentagonal fuzzy number is denoted by  $A = (a_1, a_2, a_3, a_4, a_5)$ .

Firstly, we generalize the results of addition, subtraction, multiplication, and division for two pentagonal fuzzy numbers in the following Theorem 3.2.

**THEOREM 3.2.** *Let  $A = (a_1, a_2, a_3, a_4, a_5)$  and  $B = (b_1, b_2, b_3, b_4, b_5)$  are two pentagonal fuzzy numbers. Then the results of four operations are as follows;*

1.  $A(+)B$  is a pentagonal fuzzy number.
2.  $A(-)B$  is a pentagonal fuzzy number.
3.  $A(\cdot)B$  is a fuzzy number on  $(a_1b_1, a_5b_5)$ , but doesn't need to be a pentagonal fuzzy number.
4.  $A(/)B$  is a fuzzy number on  $(\frac{a_1}{b_5}, \frac{a_5}{b_1})$ , but doesn't need to be a pentagonal fuzzy number.

*Proof.* Note that

$$\mu_A(x) = \begin{cases} 0, & x < a_1, a_5 \leq x \\ \frac{x-a_1}{2(a_2-a_1)}, & a_1 \leq x < a_2 \\ \frac{x-a_3}{2(a_3-a_2)} + 1, & a_2 \leq x < a_3 \\ \frac{a_3-x}{2(a_4-a_3)} + 1, & a_3 \leq x < a_4 \\ \frac{a_5-x}{2(a_5-a_4)}, & a_4 \leq x < a_5 \end{cases}$$

and

$$\mu_B(x) = \begin{cases} 0, & x < b_1, b_5 \leq x \\ \frac{x-b_1}{2(b_2-b_1)}, & b_1 \leq x < b_2 \\ \frac{x-b_3}{2(b_3-b_2)} + 1, & b_2 \leq x < b_3 \\ \frac{b_3-x}{2(b_4-b_3)} + 1, & b_3 \leq x < b_4 \\ \frac{b_5-x}{2(b_5-b_4)}, & b_4 \leq x < b_5 \end{cases}$$

where  $a_i, b_i \in \mathbb{R}, i = 1, 2, 3, 4, 5, a_1 + a_3 > 2a_2, a_3 + a_5 < 2a_4, b_1 + b_3 > 2b_2,$  and  $b_3 + b_5 < 2b_4.$

Four operations are calculated by using  $\alpha$ -cuts. Let  $A_\alpha = [a_1^{(\alpha)}, a_2^{(\alpha)}]$  and  $B_\alpha = [b_1^{(\alpha)}, b_2^{(\alpha)}]$  are the  $\alpha$ -cuts of  $A$  and  $B.$  If  $0 \leq \alpha < \frac{1}{2},$  we have

$$\alpha = \frac{a_1^{(\alpha)} - a_1}{2(a_2 - a_1)} \quad \text{and} \quad \alpha = \frac{a_5 - a_2^{(\alpha)}}{2(a_5 - a_4)}.$$

Thus  $A_\alpha = [a_1 + 2\alpha(a_2 - a_1), a_5 - 2\alpha(a_5 - a_4)].$  Similarly,  $B_\alpha = [b_1 + 2\alpha(b_2 - b_1), b_5 - 2\alpha(b_5 - b_4)].$  If  $\frac{1}{2} \leq \alpha < 1,$  we have

$$\alpha = \frac{a_1^{(\alpha)} - a_3}{2(a_3 - a_2)} + 1 \quad \text{and} \quad \alpha = \frac{a_3 - a_2^{(\alpha)}}{2(a_4 - a_3)} + 1.$$

Thus  $A_\alpha = [a_3 + 2(\alpha - 1)(a_3 - a_2), a_3 - 2(\alpha - 1)(a_4 - a_3)].$  Similarly,  $B_\alpha = [b_3 + 2(\alpha - 1)(b_3 - b_2), b_3 - 2(\alpha - 1)(b_4 - b_3)].$

1. Addition : For  $0 \leq \alpha < \frac{1}{2},$

$$\begin{aligned} A_\alpha(+)B_\alpha &= [a_1^{(\alpha)} + b_1^{(\alpha)}, a_2^{(\alpha)} + b_2^{(\alpha)}] \\ &= [a_1 + b_1 + 2\alpha(a_2 + b_2 - a_1 - b_1), \\ &\quad a_5 + b_5 - 2\alpha(a_5 + b_5 - a_4 - b_4)]. \end{aligned}$$

If  $x \in [a_1 + b_1, a_2 + b_2],$  then  $x = a_1 + b_1 + 2\alpha(a_2 + b_2 - a_1 - b_1).$  Thus  $\alpha = \frac{x - a_1 - b_1}{2(a_2 + b_2 - a_1 - b_1)}.$  Similarly, we obtain  $\alpha = \frac{x - a_5 - b_5}{2(a_4 + b_4 - a_5 - b_5)}$  for  $x \in [a_4 + b_4, a_5 + b_5].$  For  $\frac{1}{2} \leq \alpha < 1,$

$$\begin{aligned} A_\alpha(+)B_\alpha &= [a_1^{(\alpha)} + b_1^{(\alpha)}, a_2^{(\alpha)} + b_2^{(\alpha)}] \\ &= [a_3 + b_3 + 2(\alpha - 1)(a_3 + b_3 - a_2 - b_2), \\ &\quad a_3 + b_3 - 2(\alpha - 1)(a_4 + b_4 - a_3 - b_3)]. \end{aligned}$$

If  $x \in [a_2 + b_2, a_3 + b_3],$  then  $x = a_3 + b_3 + 2(\alpha - 1)(a_3 + b_3 - a_2 - b_2).$  Thus  $\alpha = \frac{x - a_3 - b_3}{2(a_3 + b_3 - a_2 - b_2)} + 1.$  Similarly, we obtain  $\alpha = \frac{x - a_3 - b_3}{2(a_3 + b_3 - a_4 - b_4)} + 1$  for  $x \in [a_3 + b_3, a_4 + b_4].$  Thus

$$\mu_{A(+)B}(x) = \begin{cases} 0, & x < a_1 + b_1, a_5 + b_5 \leq x \\ \frac{x - a_1 - b_1}{2(a_2 + b_2 - a_1 - b_1)}, & a_1 + b_1 \leq x < a_2 + b_2 \\ \frac{x - a_3 - b_3}{2(a_3 + b_3 - a_2 - b_2)} + 1, & a_2 + b_2 \leq x < a_3 + b_3 \\ \frac{x - a_3 - b_3}{2(a_3 + b_3 - a_4 - b_4)} + 1, & a_3 + b_3 \leq x < a_4 + b_4 \\ \frac{x - a_5 - b_5}{2(a_4 + b_4 - a_5 - b_5)}, & a_4 + b_4 \leq x < a_5 + b_5 \end{cases}$$

Since  $a_1 + a_3 > 2a_2$ ,  $a_3 + a_5 < 2a_4$ ,  $b_1 + b_3 > 2b_2$ , and  $b_3 + b_5 < 2b_4$ , we have  $(a_1 + b_1) + (a_3 + b_3) > 2(a_2 + b_2)$  and  $(a_3 + b_3) + (a_5 + b_5) < 2(a_4 + b_4)$ . Thus  $A(+ )B = (a_1 + b_1, a_2 + b_2, a_3 + b_3, a_4 + b_4, a_5 + b_5)$  is a pentagonal fuzzy number.

2. Subtraction : For  $0 \leq \alpha < \frac{1}{2}$ ,

$$\begin{aligned} A_\alpha(-)B_\alpha &= [a_1^{(\alpha)} - b_2^{(\alpha)}, a_2^{(\alpha)} - b_1^{(\alpha)}] \\ &= [a_1 - b_5 + 2\alpha(a_2 + b_5 - a_1 - b_4), \\ &\quad a_5 - b_1 - 2\alpha(a_5 + b_2 - a_4 - b_1)]. \end{aligned}$$

If  $x \in [a_1 - b_5, a_2 - b_4]$ , then  $x = a_1 - b_5 + 2\alpha(a_2 + b_5 - a_1 - b_4)$ . Thus  $\alpha = \frac{x - a_1 + b_5}{2(a_2 + b_5 - a_1 - b_4)}$ . Similarly, we obtain  $\alpha = \frac{x - a_5 + b_1}{2(a_4 + b_1 - a_5 - b_2)}$  for  $x \in [a_4 - b_2, a_5 - b_1]$ . For  $\frac{1}{2} \leq \alpha < 1$ ,

$$\begin{aligned} A_\alpha(-)B_\alpha &= [a_1^{(\alpha)} - b_2^{(\alpha)}, a_2^{(\alpha)} - b_1^{(\alpha)}] \\ &= [a_3 - b_3 + 2(\alpha - 1)(a_3 + b_4 - a_2 - b_3), \\ &\quad a_3 - b_3 - 2(\alpha - 1)(a_4 + b_3 - a_3 - b_2)]. \end{aligned}$$

If  $x \in [a_2 - b_4, a_3 - b_3]$ , then  $x = a_3 - b_3 + 2(\alpha - 1)(a_3 + b_4 - a_2 - b_3)$ . Thus  $\alpha = \frac{x - a_3 + b_3}{2(a_3 + b_4 - a_2 - b_3)} + 1$ . Similarly, we obtain  $\alpha = \frac{x - a_3 + b_3}{2(a_3 + b_2 - a_4 - b_3)} + 1$  for  $x \in [a_3 - b_3, a_4 - b_2]$ . Thus

$$\mu_{A(-)B}(x) = \begin{cases} 0, & x < a_1 - b_5, a_5 - b_1 \leq x \\ \frac{x - a_1 + b_5}{2(a_2 + b_5 - a_1 - b_4)}, & a_1 - b_5 \leq x < a_2 - b_4 \\ \frac{x - a_3 + b_3}{2(a_3 + b_4 - a_2 - b_3)} + 1, & a_2 - b_4 \leq x < a_3 - b_3 \\ \frac{x - a_3 + b_3}{2(a_3 + b_2 - a_4 - b_3)} + 1, & a_3 - b_3 \leq x < a_4 - b_2 \\ \frac{x - a_5 + b_1}{2(a_4 + b_1 - a_5 - b_2)}, & a_4 - b_2 \leq x < a_5 - b_1 \end{cases}$$

Since  $a_1 + a_3 > 2a_2$ ,  $a_3 + a_5 < 2a_4$ ,  $b_1 + b_3 > 2b_2$ , and  $b_3 + b_5 < 2b_4$ , we have  $(a_1 - b_5) + (a_3 - b_3) > 2(a_2 - b_4)$  and  $(a_3 - b_3) + (a_5 - b_1) < 2(a_4 - b_2)$ . Thus  $A(-)B = (a_1 - b_5, a_2 - b_4, a_3 - b_3, a_4 - b_2, a_5 - b_1)$  is a pentagonal fuzzy number.

3. Multiplication : For  $0 \leq \alpha < \frac{1}{2}$ ,

$$\begin{aligned} A_\alpha(\cdot)B_\alpha &= [a_1^{(\alpha)}b_1^{(\alpha)}, a_2^{(\alpha)}b_2^{(\alpha)}] \\ &= [\{a_1 + 2\alpha(a_2 - a_1)\}\{b_1 + 2\alpha(b_2 - b_1)\}, \\ &\quad \{a_5 - 2\alpha(a_5 - a_4)\}\{b_5 - 2\alpha(b_5 - b_4)\}]. \end{aligned}$$

If  $x \in [a_1b_1, a_2b_2]$ , then  $x = \{a_1 + 2\alpha(a_2 - a_1)\}\{b_1 + 2\alpha(b_2 - b_1)\}$ . Thus

$$\alpha = \frac{2a_1b_1 - a_2b_1 - a_1b_2 \pm \sqrt{f_1(x)}}{4(a_1b_1 - a_2b_1 - a_1b_2 + a_2b_2)},$$

where  $f_1(x) = a_2^2b_1^2 - 2a_1a_2b_1b_2 + a_1^2b_2^2 + 4a_1b_1x - 4a_2b_1x - 4a_1b_2x + 4a_2b_2x$ . Similarly, we obtain

$$\alpha = \frac{2a_5b_5 - a_5b_4 - a_4b_5 \pm \sqrt{f_2(x)}}{4(a_4b_4 - a_5b_4 - a_4b_5 + a_5b_5)},$$

where  $f_2(x) = a_5^2b_4^2 - 2a_4a_5b_4b_5 + a_4^2b_5^2 + 4a_4b_4x - 4a_5b_4x - 4a_4b_5x + 4a_5b_5x$  for  $x \in [a_4b_4, a_5b_5]$ . For  $\frac{1}{2} \leq \alpha < 1$ ,

$$A_\alpha(\cdot)B_\alpha = [\{a_3 + 2(\alpha - 1)(a_3 - a_2)\}\{b_3 + 2(\alpha - 1)(b_3 - b_2)\}, \\ \{a_3 - 2(\alpha - 1)(a_4 - a_3)\}\{b_3 - 2(\alpha - 1)(b_4 - b_3)\}].$$

If  $x \in [a_2b_2, a_3b_3]$ , then  $x = \{a_3 + 2(\alpha - 1)(a_3 - a_2)\}\{b_3 + 2(\alpha - 1)(b_3 - b_2)\}$ . Thus

$$\alpha = \frac{4a_2b_2 - 3a_3b_2 - 3a_2b_3 + 2a_3b_3 \pm \sqrt{g_1(x)}}{4(a_2b_2 - a_3b_2 - a_2b_3 + a_3b_3)},$$

where  $g_1(x) = a_3^2b_2^2 - 2a_2a_3b_2b_3 + a_2^2b_3^2 + 4a_2b_2x - 4a_3b_2x - 4a_2b_3x + 4a_3b_3x$ . Similarly, we obtain

$$\alpha = \frac{2a_3b_3 - 3a_4b_3 - 3a_3b_4 + 4a_4b_4 \pm \sqrt{g_2(x)}}{4(a_3b_3 - a_4b_3 - a_3b_4 + a_4b_4)},$$

where  $g_2(x) = a_4^2b_3^2 - 2a_3a_4b_3b_4 + a_3^2b_4^2 + 4a_3b_3x - 4a_4b_3x - 4a_3b_4x + 4a_4b_4x$  for  $x \in [a_3b_3, a_4b_4]$ . Thus

$$\mu_{A(\cdot)B}(x) = \begin{cases} 0, & x < a_1b_1, a_5b_5 \leq x \\ \frac{2a_1b_1 - a_2b_1 - a_1b_2 + \sqrt{f_1(x)}}{4(a_1b_1 - a_2b_1 - a_1b_2 + a_2b_2)}, & a_1b_1 \leq x < a_2b_2 \\ \frac{4a_2b_2 - 3a_3b_2 - 3a_2b_3 + 2a_3b_3 + \sqrt{g_1(x)}}{4(a_2b_2 - a_3b_2 - a_2b_3 + a_3b_3)}, & a_2b_2 \leq x < a_3b_3 \\ \frac{2a_3b_3 - 3a_4b_3 - 3a_3b_4 + 4a_4b_4 - \sqrt{g_2(x)}}{4(a_3b_3 - a_4b_3 - a_3b_4 + a_4b_4)}, & a_3b_3 \leq x < a_4b_4 \\ \frac{2a_5b_5 - a_5b_4 - a_4b_5 - \sqrt{f_2(x)}}{4(a_4b_4 - a_5b_4 - a_4b_5 + a_5b_5)}, & a_4b_4 \leq x < a_5b_5 \end{cases}$$

Hence  $A(\cdot)B$  is a fuzzy set on  $(a_1b_1, a_5b_5)$ , but doesn't need to be a quadrangular fuzzy set or a pentagonal fuzzy number.

4. Division : For  $0 \leq \alpha < \frac{1}{2}$ ,

$$A_\alpha(/)B_\alpha = [a_1^{(\alpha)} / b_2^{(\alpha)}, a_2^{(\alpha)} / b_1^{(\alpha)}] \\ = \left[ \frac{a_1 + 2\alpha(a_2 - a_1)}{b_5 - 2\alpha(b_5 - b_4)}, \frac{a_5 - 2\alpha(a_5 - a_4)}{b_1 + 2\alpha(b_2 - b_1)} \right].$$

If  $x \in [a_1/b_5, a_2/b_4]$ , then  $x = \frac{a_1+2\alpha(a_2-a_1)}{b_5-2\alpha(b_5-b_4)}$ . Thus

$$\alpha = \frac{a_1 - b_5x}{2(a_1 - a_2 + b_4x - b_5x)}.$$

Similarly, we obtain

$$\alpha = -\frac{a_5 - b_1x}{2(a_4 - a_5 + b_1x - b_2x)}$$

for  $x \in [a_4/b_2, a_5/b_1]$ . For  $\frac{1}{2} \leq \alpha < 1$ ,

$$A_\alpha(/)B_\alpha = \left[ \frac{a_3 + 2(\alpha - 1)(a_3 - a_2)}{b_3 - 2(\alpha - 1)(b_4 - b_3)}, \frac{a_3 - 2(\alpha - 1)(a_4 - a_3)}{b_3 + 2(\alpha - 1)(b_3 - b_2)} \right].$$

If  $x \in [a_2/b_4, a_3/b_3]$ , then  $x = \frac{a_3+2(\alpha-1)(a_3-a_2)}{b_3-2(\alpha-1)(b_4-b_3)}$ . Thus

$$\alpha = \frac{2a_2 - a_3 + b_3x - 2b_4x}{2(a_2 - a_3 + b_3x - b_4x)}.$$

Similarly, we obtain

$$\alpha = \frac{a_3 - 2a_4 + 2b_2x - b_3x}{2(a_3 - a_4 + b_2x - b_3x)}$$

for  $x \in [a_3/b_3, a_4/b_2]$ . Thus

$$\mu_{A(/)B}(x) = \begin{cases} 0, & x < a_1/b_5, a_5/b_1 \leq x \\ \frac{a_1 - b_5x}{2(a_1 - a_2 + b_4x - b_5x)}, & a_1/b_5 \leq x < a_2/b_4 \\ \frac{2a_2 - a_3 + b_3x - 2b_4x}{2(a_2 - a_3 + b_3x - b_4x)}, & a_2/b_4 \leq x < a_3/b_3 \\ \frac{a_3 - 2a_4 + 2b_2x - b_3x}{2(a_3 - a_4 + b_2x - b_3x)}, & a_3/b_3 \leq x < a_4/b_2 \\ -\frac{a_5 - b_1x}{2(a_4 - a_5 + b_1x - b_2x)}, & a_4/b_2 \leq x < a_5/b_1 \end{cases}$$

Hence  $A(/)B$  is a fuzzy set on  $(\frac{a_1}{b_5}, \frac{a_5}{b_1})$ , but doesn't need to be a quadrangular fuzzy set or a pentagonal fuzzy number. □

REMARK 3.3. According to Remark 3.3 and Remark 3.6 in [2], given a left(right) quadrangular fuzzy set  $A$ , there is a left(right) quadrangular fuzzy set  $B$  such that  $A(+)B$  becomes a triangular fuzzy number. But, in this case, there does not exist a left(right) quadrangular fuzzy set  $B$  such that  $A(+)B$  becomes a triangular fuzzy number or a quadrangular fuzzy set since the definition of a left(right) quadrangular fuzzy set is different. Similarly, by Remark 3.8 in [2], given a left(right) quadrangular fuzzy set  $A$ , there is a right(left) quadrangular fuzzy set  $B$  such that  $A(-)B$  becomes a triangular fuzzy number. But, in this case, there does not exist a left(right) quadrangular fuzzy set  $B$  such that  $A(-)B$  becomes a



triangular fuzzy number or a quadrangular fuzzy set since the definition of a left(right) quadrangular fuzzy set is different.

EXAMPLE 3.4. Let  $A = (1, 2, 5, 7, 8)$  and  $B = (2, 4, 7, 10, 11)$ . Then we have the followings.

1. The extended addition reduces to

$$\mu_{A(+ )B}(x) = \begin{cases} 0 & (x < 3, 19 \leq x) \\ (-3 + x)/6 & (3 \leq x < 6) \\ x/12 & (6 \leq x < 12) \\ (22 - x)/10 & (12 \leq x < 17) \\ (19 - x)/4 & (17 \leq x < 19) \end{cases}$$

2. The extended subtraction reduces to

$$\mu_{A(- )B}(x) = \begin{cases} 0 & (x < -10, 6 \leq x) \\ (10 + x)/4 & (-10 \leq x < -8) \\ (14 + x)/12 & (-8 \leq x < -2) \\ (8 - x)/10 & (-2 \leq x < 3) \\ (6 - x)/6 & (3 \leq x < 6) \end{cases}$$

3. The extended multiplication reduces to

$$\mu_{A(\cdot )B}(x) = \begin{cases} 0 & (x < 2, 88 \leq x) \\ (-4 + 2\sqrt{2x})/8 & (2 \leq x < 8) \\ \sqrt{36 + 36x}/36 & (8 \leq x < 35) \\ (53 - \sqrt{1 + 24x})/24 & (35 \leq x < 70) \\ (19 - \sqrt{9 + 4x})/4 & (70 \leq x < 88) \end{cases}$$

4. The extended division reduces to

$$\mu_{A(/ )B}(x) = \begin{cases} 0 & (x < \frac{1}{11}, 4 \leq x) \\ (1 - 11x)/2(-1 - x) & (\frac{1}{11} \leq x < \frac{1}{5}) \\ (1 + 13x)/2(3 + 3x) & (\frac{1}{5} \leq x < \frac{5}{7}) \\ (-9 + x)/2(-2 - 3x) & (\frac{5}{7} \leq x < \frac{7}{4}) \\ (8 - 2x)/2(1 + 2x) & (\frac{7}{4} \leq x < 4). \end{cases}$$

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